Understanding DSGE

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Para Dair, Priscilla e Manuela:
obrigado pela paciência, amor e apoio.

Preface

When one resolves to write a book, and hopes that it will be well accepted, one issue arises that should definitely be addressed: “What makes this book different from those that have gone before?” This preface seeks to answer this question.

Dynamic Stochastic General Equilibrium (DSGE) models have become a point of reference in modern macroeconomics. What currently makes this methodology so important is its ability to answer any question regarding the behavior of a particular economic phenomenon.

If, on the one hand, the theoretical development of DSGE models is not overly complex to understand, on the other, their practical application is rather more difficult. The literature on this subject presents important, yet obscure points, which are difficult to comprehend. Generally, articles begin with a presentation of the agents' object functions and of the equations that solve the maximization problem, while their resolution is not shown. In many cases, it is difficult to identify both the exact theoretical model and its application. Thus, the most important part of this type of exercise is overcoming these barriers of obscurity.

Although this methodology has become so popular in the current economic literature, there is no manual that reveals, step-by-step, how this "black box" works. This deficiency poses an important challenge, as many young researchers give up this line of research on account of the initial difficulty.

Some books, although not manuals as such, aid in understanding this methodology. Wickens's "Macroeconomic Theory: A Dynamic General Equilibrium Approach" (2011) presents a view of modern macroeconomics that seeks to integrate macroeconomics and microeconomics. It is firmly rooted in general equilibrium models and demonstrates an understanding of the changes that macroeconomic methods are facing. The following four books follow practically the same logic. They begin with a basic model and, as one progresses through the book, several types of friction are incorporated. The books are: "Computational Macroeconomics for the Open Economy" by Lim and McNelis (2008); "The ABCs of RBCs" by McCandless (2008); "Monetary Policy, Inflation, and the Business Cycle" by Galí (2008); and "Introduction to Dynamic Macroeconomic General

Equilibrium Models" by Torres (2014). There are a further two books that deal with the methodology's "behind the scenes" aspects: "Structural Macroeconometrics" by DeJong and Dave (2007), and "Methods for Applied Macroeconomic Research" by Canova (2007).

In short, this work takes the best bits from each of the aforementioned books: Lim and McNelis (2008) – organization; McCandless (2008) – presentation of log-linearization and solutions; and Torres (2014) – educational methodology, in a quest for tools that are useful for overcoming initial obstacles to the study of DSGE modeling and persuading young researchers to work with this methodology. In principle, this is not a macroeconomics book per se, but one that presents the tools used in the development of these models. The idea is that it acts as a complement to the books mentioned in the previous paragraph while at the same time presenting the models in greater detail, offering a step-by-step course. The target audiences are advanced undergraduate students, graduate students and experienced economists prepared to learn this methodology. The book begins with a basic Real Business Cycle model and, gradually, the frictions of a standard DSGE model are incorporated: imperfect competition, price and wage frictions, habit formation, non-Ricardian agents, investment adjustment costs, capacity underutilization costs, and lastly, government.

Chapter 2

Real Business Cycle (RBC) model

This chapter presents a simple Real Business Cycle (RBC) model, assuming perfect competition and fully flexible prices in all markets.

Real business cycle theory states that supply shocks (technological shocks) are what generate economic fluctuations, and uses a neoclassical growth model as a reference for the economy's long-term behavior. The model's basic structure is relatively simple. It describes the behavior of two types of agent: households and firms. In practice, there is a very large number of households that are treated as if they were identical. Thus, one may use the term "representative household" or simply "household". As for firms, the same logic applies: there is a large number of firms, however, they possess the same technology and can thus be typified as a representative firm. It is appropriate to mention that this type of model is not limited to these two economic agents (households and firms). A "complete" model would consist of five agents: households, firms, fiscal and monetary authorities, the foreign sector and financial institutions.

Brief theoretical review: Real Business Cycles

In order to present the basic ideas involved in this type of model, this section demonstrates how households solve two problems of choice: intratemporal consumption-leisure and intertemporal consumption-savings. It also deals with how firms choose the inputs used in the production process. Basically, in all cases, the marginal rate of substitution is compared to relative price. For the first problem, a model with two goods (consumption and leisure), and how optimal choice occurs within this trade-off, will be analyzed. The second problem will be presented using a simple two-period intertemporal model. Lastly, a firm's profit maximization problem is demonstrated. The ideas presented in this section are simple, but suffice to demonstrate how these choices are made within this model and in the rest of the book.

Model with two "goods": consumption and leisure

In this initial study of consumer theory, it will be assumed that there are two large categories of consumer good, "good 1" and "good 2". Because of to the interest in studying how consumers choose what they consume, one must define how these agents earn their income. The most obvious way is to think that consumers obtain income from their labor. Thus, an individual may choose to work a certain number of hours, receiving wage W per hour. Presumably, work is a consumer "bad", that is, agents do not like to work because the more they do, the less time they have for leisure. Thus, with the aim of adapting the model to standard consumer theory, instead of considering a "bad" (work), one must consider a "good" (leisure), defined as the number of hours left after subtracting the time spent working from the total number of available hours in a certain period[1].

Indifference curves (consumption-leisure)

The two factors that provide an individual with utility are consumption (C) and leisure, $u(C, leisure)$. Here, both consumption and leisure will be treated as goods, even though leisure is somewhat intangible[2]. Initially, it is useful to think of the general properties of a utility function $u(C, leisure)$ as the standard properties of consumer theory. Thus, the utility function is assumed to have the following properties:

[1] Leisure + Work = total available time.

[2] Leisure can be analyzed as a good, being a function of the opportunity cost, availability and preferences. The question then arises, "what is the opportunity cost of leisure?" The cost of spending hours watching television is basically the amount of money that one would receive if one were working instead. Therefore, the opportunity cost of one hour of leisure should be the same as the wage for one hour's work. Availability is directly related to the amount of household income and preference is related to a household's sensitivity to demand for leisure given changes in income or wages (Ehrenberg and Smith, 2000).

1. Strictly increasing, $\frac{\partial u}{\partial C} > 0$ and $\frac{\partial u}{\partial leisure} > 0$; and

2. Diminishing marginal returns, $\frac{\partial^2 u}{\partial C^2} < 0$ and $\frac{\partial^2 u}{\partial leisure^2} < 0$.

Definition 2.1.1 (Indifference curve)**.** *An indifference curve shows a grouping of consumer bundles for which an individual is indifferent. In other words, all bundles provide the same utility.*

With these assumptions, it is possible to plot an indifference curve map for consumption and leisure, as illustrated in Figure 2.1. Each indifference curve possesses the standard properties of consumer theory. Specifically, each curve has a negative slope, is convex to the origin and may not cross another indifference curve.

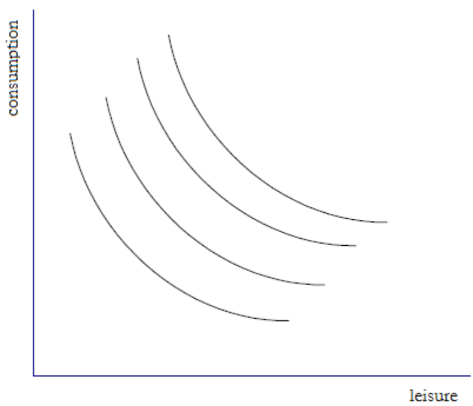

Figure 2.1: Indifference curve map for consumption and leisure.

Although these two goods (consumption and leisure) are not on the goods market (one cannot really buy leisure), there is still a well-defined idea of a ”marginal rate of substitution” (MRS) between them. The MRS measures how many units of a good one is willing to give up in exchange for another good. On a graph, the MRS is the slope of the indifference curve.

Definition 2.1.2 (Marginal Rate of Substitution)**.** *The negative slope of an indifference curve of a bundle formed by two goods, X and Y, is referred to as the Marginal Rate of Substitution (MRS) at that point. That is,*

$$MRS_{X,Y} = -\frac{\partial Y}{\partial X}\bigg|_{U=U_1} = -\frac{MU_X}{MU_Y}\bigg|_{U=U_i}$$

where MU_X and MU_Y represent the marginal utilities in relation to goods X and Y, respectively, and the $|_{U=U_i}$ notation indicates that the slope is calculated along the indifference curve U_i.

In short, in the consumption-leisure model, the marginal rate of substitution of leisure with consumption, represented by $MRS_{\text{Leisure},C}$, is the rate at which a consumer is willing to give up leisure for consumer goods.

Budget constraints

An indifference map is not sufficient to study a consumer's optimal choice. To this end, the individual's budget constraint is required. Here, the amount of income an individual has to spend on consumption depends on how much he/she chooses to work. For the purposes of this study on budget constraint, suppose an individual has 60 hours available per week for work and leisure[3].

Assuming that an individual can work the amount of hours he/she likes (L), receiving an hourly wage W, total weekly income is:

$$Y = LW$$

[3]Of the 168 hours (24 x 7) in a week, the weekends and hours intended for the individual's subsistence (bathing, meals, etc.) are being subtracted. So the number of daily and weekly hours available for work-leisure are 12 hours and 60 hours (12 x 5) respectively.

As previously mentioned, the number of work hours (L) plus the number of leisure hours per week, must be equal to 60 hours, $L = 60 - leisure$. Thus, income can be written as a function of leisure:

$$Y = (60 - leisure)W$$

Another simplifying assumption is that individuals spend all their income on consumption, not saving anything. Each consumer good, c, can be bought on the market for the price P. Thus, an individual's consumption in each period is:

$$Pc = Y$$

Combining the two previous expressions, we arrive at the following budget constraint:

$$Pc = (60 - leisure)W$$

In this expression, an individual takes the prices of consumer goods (P) and hourly wages (W) as a given, choosing the level of consumption and amount of leisure. Rearranging the previous budget constraint,

$$\overbrace{\underbrace{Pc}_{\text{consumer goods}} + \underbrace{Wleisure}_{\text{leisure}}}^{\text{destination of income}} = \underbrace{60W}_{\text{total disposable income}}$$

it can be seen that the period's total disposable income ($60W$) is used to acquire consumer goods (Pc) and leisure ($Wleisure$). As mentioned above, leisure is not directly bought or sold on the market. However, wages are the opportunity cost of leisure; each hour spent on leisure is an hour that could have been spent working. Thus, from an economic point of view, in which opportunity costs are explicitly considered, wages are the price of leisure.

A budget constraint describes the set of choices available to a consumer, but reveals nothing about the choice to be made within this set. To plot the budget constraint on a graph, as in Figure 2.1, the equation must be rearranged in the following way:

$$c = \left(\frac{60W}{P}\right) - \left(\frac{W}{P}\right) leisure$$

The budget constraint is a line with a vertical intercept of $\left(\frac{60W}{P}\right)$ and slope $-\left(\frac{W}{P}\right)$. When $c = 0$ the horizontal intercept is $leisure = 60$, showing that if an individual does not want to consume any goods, he/she will use all of his/her time for leisure.

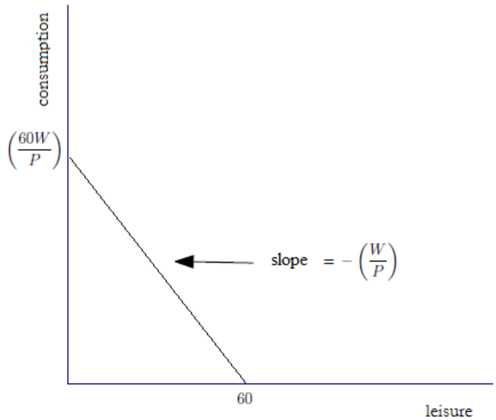

Figure 2.2: Budget constraint line for the consumption-leisure model..

Individuals' decisions regarding consumption and work

To obtain optimal choice, the interaction of individuals' preferences (indifference curve maps) with their budget constraints must be considered. Formally, an individual's problem is:

$$\max_{c,L} u(c, L)$$

subject to,

$$Pc = WL$$

Many optimization problems can be solved using the Lagrangian method:

$$\mathscr{L} = u(c, L) - \lambda(Pc - WL)$$

With the following first-order conditions:

$$\frac{\partial \mathscr{L}}{\partial c} = \frac{\partial u}{\partial c} - \lambda P = 0$$

$$\frac{\partial \mathscr{L}}{\partial L} = \frac{\partial u}{\partial L} + \lambda W = 0$$

Combining the two previous expressions:

Theoretical result 2.1.1 (Supply of Labor)**.**

$$\underbrace{\frac{\partial u/\partial L}{\partial u/\partial c}}_{\text{MRS L-c}} = \underbrace{-\frac{W}{P}}_{\text{Relative price L-c}}$$

On a graph, at point E (figure 2.3), leisure-consumption's marginal rate of substitution is equal to leisure-consumption's relative price[4]. On the other hand, at point D, leisure-consumption's relative price ($\frac{W}{P}$) exceeds leisure-consumption's marginal rate of substitution. If this occurs, an individual will be better off working more (enjoying less leisure) and using the additional income to expand consumption. Thus, with the increased acquisition of consumer goods, leisure-consumption's MRS increases. When this initial difference ceases to exist (point E), there is no more incentive for an individual to increase his/her level of work. In other words, the leisure-consumption bundle represented by point D belongs to indifference curve U_1, following the budget constraint line towards point E. It should be noted that, of all the points on the budget constraint line, it is this point that is tangential to the highest indifference curve (U_2). Therefore, given his/her budget constraint, the individual will be in a better situation at point E than at point D.

[4]The reader must remember that, for graphical analysis, it is better to use the consumption-leisure instead of the consumption-work locus, as the first pair represents two "goods", whereas the second pair represents a "good" and a "bad".

Definition 2.1.3 (The problem of the household)**.** *To maximize utility, given a fixed amount of income, an individual will buy the amount of goods that depletes his/her total income equating to the physical rate of tradeoff between any two goods (MRS) and the rate at which a good can be exchanged for another on the market (relative price).*

Definition 2.1.4 (Optimal result of the problem of the household)**.** *The optimal consumption bundle is the point that represents the pair of goods that is on the highest indifference curve and is within the individual's budget constraint.*

In summary, each individual chooses a combination of consumption and leisure that maximizes his/her utility. Thus, an individual chooses the pair $(C^*, leisure^*)$ (figure 2.3) for which the budget constraint is the tangent of an indifference curve.

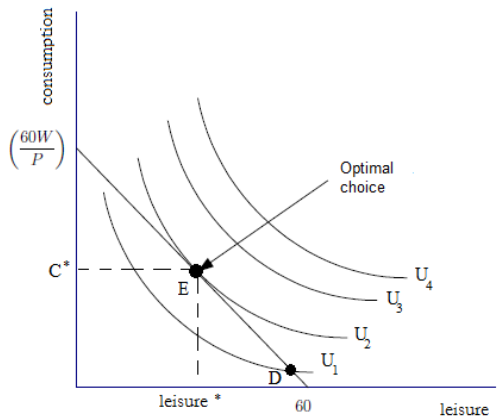

Figure 2.3: Optimal choice of consumption and leisure.

Labor supply function

When an individual optimally chooses to spend H hours of his/her time on leisure then, at the same time, he/she is choosing to spend $L = 60 - H$ hours of his/her time working. Therefore, the individual is supplying L hours of work to this market. Evidently, the choice of the amount of labor in figure 2.3 depends on the level of wages (W). Thus, the budget constraint is $c = 60w - wH$, where $w = \frac{W}{P}$ is the real wage.

Initially, it will be assumed that real wages are at a very low level (w_1). At this starting point, the optimal choice will be point A (figure 2.4). This choice is associated with the amount of labor L_1. Now suppose that real wages increase, $w_2 > w_1$. The new optimal choice is point B. At this point, the individual enjoys less leisure compared to point A ($L_2 > L_1$).

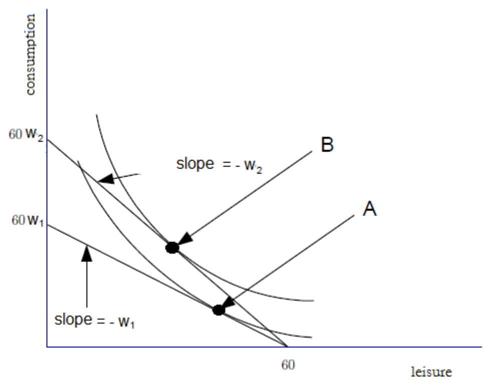

Figure 2.4: With a rise in real wages from w_1 to w_2, the individual chooses more consumption and less leisure.

Now suppose that the real wage increases to w_3. The optimal choice at this new real wage level is at point C (figure 2.5). Comparing point C to point B, the individual does not adjust his/her amount of labor hours when real wages rise from w_2 to w_3. Thus, at this wage level, the individual works L_3 hours, with $L_3 = L_2 > L_1$.

Consider yet another rise in real wages ($w_4 > w_3$).). At this point, real wages are high enough that the individual does not increase his/her amount of work to keep the same level of consumption. At this level of wages, it is reasonable to expect that the indi-

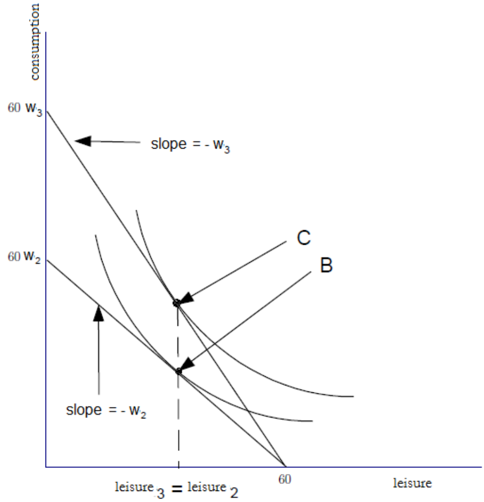

Figure 2.5: With a rise in real wages from w_2 to w_3, the individual chooses more consumption and the same amount of leisure.

vidual chooses to spend less time working and more time on leisure ($L_4 < L_3$). In figure 2.6, an increase in wages causes the optimal choice to move from point C to point D. At this point, the individual works fewer hours than at point C.

Substitution and income effects

The effects of changes in real wages on optimal leisure choice can be separated into two components: a substitution effect and an income effect. Both effects have a general significance within economics and can indeed be applied to any optimal choice problem.

In the context of this consumption-leisure model, the substitution effect of higher real wages leads the individual to choose less leisure (work more). In other words, because of to the higher level of wages, the opportunity cost of leisure has risen. Thus, the individual would tend to demand less leisure. Conversely, the income effect of higher real wages causes the individual to choose more leisure (less work). That is, because of to the higher income that a higher level of real wages affords, the individual would choose

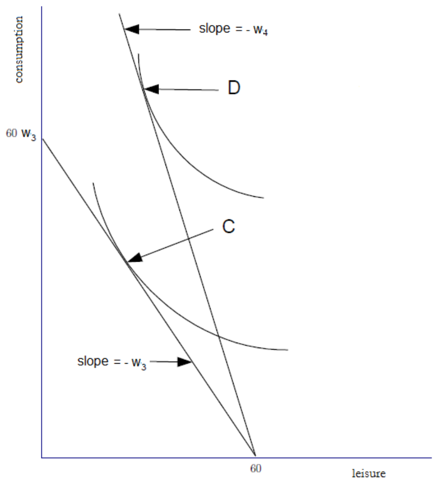

Figure 2.6: With a rise in real wages from w_3 to w_4, the individual chooses more consumption and more leisure.

a higher level of consumption of all normal goods. Assuming that leisure is a normal good, a rise in income would cause the individual to choose more leisure and thus spend less time working.

Both effects are ever present: either the substitution effect dominates the income effect (because of being stronger) and the rise in real wages causes the individual to choose more work (less leisure), or the income effect is dominant and a rise in real wages causes the individual to choose less work (more leisure), or they cancel each other out.

With this notion of income and substitution effects, the effects shown in Figures 2.4-2.6 must be reconsidered. A rise in real wages from w_1 to w_2 causes the individual to work more, as illustrated by the optimal choice moving from point A to point B. This is the section at which the substitution effect outweighs the income effect. When real wages rise from w_2 to w_3, the individual decides not to adjust the amount of work, keeping the same level of leisure. The section between points B and C corresponds to the region at which the effects exactly cancel each other out. Lastly, when wages rise from w_3 to w_4, the individual decides to work less, as shown by the optimal choice moving from point C to point D. So, this is the section at which the income effect outweighs the substitution effect (figure 2.7).

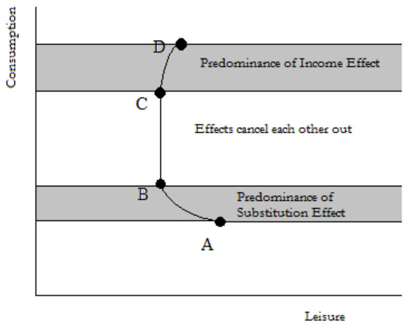

Figure 2.7: Sections at which the income and substitution effects dominate.

Dynamic structure of consumption-savings

When an individual makes his/her choice between consumption and leisure in the current period, he/she generally recognizes that a similar choice will be made in the future. This is formalized by a utility function $u(c_1, c_2, c_3, \ldots)$. Economists almost always simplify intertemporal problems assuming that preferences are additively separable, $u(c_1, c_2, c_3, \ldots) = u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \ldots$. The β parameter is called an intertemporal discount factor. Its value is less than 1 ($\beta < 1$) as it represents the fact that households are more concerned with present consumption than future consumption[5].

In this section, the aim is to assess individuals' intertemporal choices. For the sake of clarity, data regarding leisure will be ignored. It will be assumed that individuals live in two periods, the present (period 1) and the future (period 2). This division into two periods is enough to illustrate the basic principles of macroeconomic events that occur intertemporally in a structure with an infinite time horizon.

[5] $\beta = \frac{1}{1+\theta}$, where $\theta > 0$ is the subjective intertemporal preference rate. This parameter indicates the value of future utility in relation to present utility. The greater the value of β, the more patient the household is with regard to consumption.

In the intertemporal context, the two arguments that make up the utility function are period 1 consumption and period 2 consumption, which will be represented by c_1 and c_2, respectively. All the usual properties of the utility function are assumed: utility is always strictly increasing in both arguments; marginal utility always decreases in both arguments. The utility function will be written as $u(c1, c2)$, and can be represented by an indifference curve map.

In this model, individuals receive income twice during their lives - once in period 1 and once in period 2. They start off in period 1 with a certain amount of wealth, A_0. They choose consumption in period 1 (c_1) paying a price of P_1, and also decide how much wealth[6] they will carry forward to period 2, A_1. Thus, an individual's budget constraint in period 1 can be written as:

$$P_1 c_1 + A_1 = RA_0 + Y_1$$

where R is the gross nominal interest rate[7] that represents the returns on each monetary unit held as a financial asset from one period to another.

The same logic can be repeated for an individual's budget constraint in period 2:

$$P_2 c_2 + A_2 = RA_1 + Y_2$$

in which, owing to individuals living only for two periods, final wealth must be zero ($A_2 = 0$). The intertemporal representation of these events is shown as a timeline in figure 2.8.

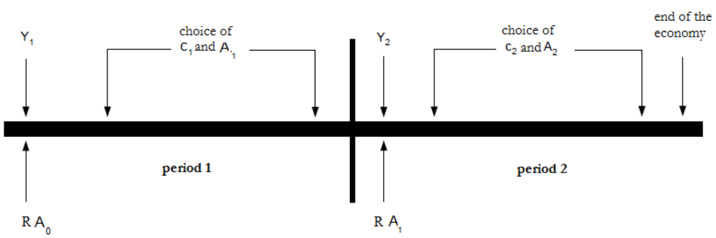

Figure 2.8: Intertemporal representation of the events of a two-period consumption structure.

[6] Note that A_0 and A_1 may take on negative values, indicating that an individual would be a borrower in these periods.

[7] A gross rate is defined as: $R = 1 + r$, where r is the net return for the period.

To continue the model, it is necessary to define a period's savings as the difference between total income and the total spent within said period:

$$S_1 = (R-1)A_0 + Y_1 - P_1c_1$$

Rearranging the period 1 budget constraint:

$$A_1 - A_0 = (R-1)A_0 + Y_1 - P_1c_1$$

Comparing the last two expressions, one can see that $S_1 = A_1 - A_0$. Thus, an individual's savings in period 1 is equal to the variation in his/her wealth within the period. Similarly, an individual's savings in period 2 is $S_2 = (R-1)A_1 + Y_2 - P_2c_2$ or $S_2 = A_2 - A_1$.

An approximation to general economic behavior is to suppose that individuals are rational during their life spans, in the sense that they save and/or borrow appropriately during their lives. Given this structure and taking the assumption of rationality into account, analysis of the model may begin. Thus, by combining the budget constraints of periods 1 and 2, we arrive at an individual's intertemporal budget constraint. Solving period 1's budget constraint for A_1:

$$A_1 = RA_0 + Y_1 - P_1c_1$$

substituting this result in period 2's budget constraint,

$$P_2c_2 = R[RA_0 + Y_1 - P_1c_1] + Y_2$$

dividing both sides of the previous expression by R:

$$P_1c_1 + \frac{P_2c_2}{R} = Y_1 + \frac{Y_2}{R} + RA_0$$

The right-hand side of this last expression represents the discounted intertemporal resource, which considers the initial wealth and an individual's lifetime income (two periods in this model). The left-hand side represents discounted intertemporal consumption, which considers the consumption in all periods. The intertemporal budget constraint that an individual rationally uses to make his/her choices in time will be drawn in a locus $c_1 - c_2$. For the sake of simplicity, it will be assumed that initial wealth is zero ($A_0 = 0$).

Solving the previous expression for c_2:

$$c_2 = \left[\left(\frac{R}{P_2}\right)Y_1 + \frac{Y_2}{P_2}\right] - \left[\frac{P_1 R}{P_2}\right]c_1$$

Thus, the vertical intercept is $\left[\left(\frac{R}{P_2}\right)Y_1 + \frac{Y_2}{P_2}\right]$ and the slope is $\left[\frac{P_1 R}{P_2}\right]$. The graph representing an individual's intertemporal budget constraint is shown in figure 2.9.

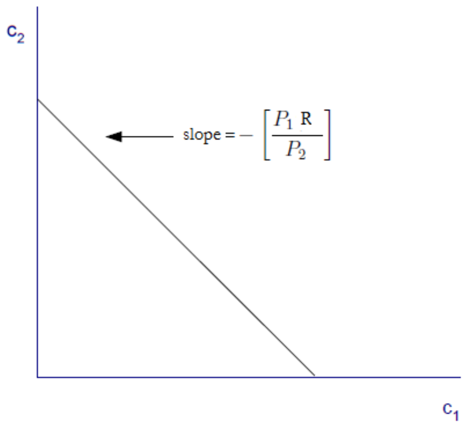

Figure 2.9: An individual's intertemporal budget constraint.

Optimal intertemporal choice

An individual's optimal intertemporal choice is an interaction between his/her indifference curve map and intertemporal budget constraints. In this model, an individual lives for two periods. In this case, his/her preferences can be reduced to:

$$u(c_1, c_2) = u(c_1) + \beta u(c_2)$$

Given that the individual will not consume in period 3, it can be assumed that keeping assets in the form of savings in period 2 would not be optimal ($A_2 = 0$). Thus, an individual's budget constraints in both periods are:

$$P_1 c_1 + A_1 = RA_0 + Y_1$$

$$P_2 c_2 = RA_1 + Y_2$$

The problem for the individual is to choose the levels of consumption for both periods, c_1 and c_2, and the level of wealth A_1 that maximizes his/her utility function, which is subject to budget constraints in both periods. The values of P and R are given. Thus, the problem of the individual can be written as:

$$\max_{c_1, c_2, A_1} u(c_1) + \beta u(c_2)$$

subject to,

$$P_1 c_1 + A_1 = RA_0 + Y_1$$
$$P_2 c_2 = RA_1 + Y_2$$

The Lagrangian for this problem is:

$$\mathscr{L} = u(c_1) + \beta u(c_2) - \lambda_1 [P_1 c_1 + A_1 - RA_0 - Y_1] - \lambda_2 [P_2 c_2 - RA_1 - Y_2]$$

The problem's first-order conditions are:

$$\frac{\partial \mathscr{L}}{\partial c_1} = \frac{\partial u}{\partial c_1} - \lambda_1 P_1 = 0$$

$$\frac{\partial \mathscr{L}}{\partial c_2} = \beta \frac{\partial u}{\partial c_2} - \lambda_2 P_2 = 0$$

$$\frac{\partial \mathscr{L}}{\partial A_1} = -\lambda_1 + \lambda_2 R = 0$$

Rewriting the first two first-order conditions, $\lambda_1 = \frac{\partial u/\partial c_1}{P_1}$ and $\lambda_2 = \beta \frac{\partial u/\partial c_2}{P_2}$, substituting these values in the third first-order condition and defining $\pi_2 = \frac{P_2}{P_1}$:

Theoretical result 2.1.2 (Euler Equation)**.**

$$\underbrace{-\frac{\partial u/\partial c_1}{\beta \partial u/\partial c_2}}_{MRS\ c_1\text{-}c_2} = \underbrace{-\frac{R}{\pi_2}}_{Relative\ price\ c_1\text{-}c_2}$$

This is called a Euler Equation. It relates the marginal utility of consumption for both periods (MRS c_1-c_2) with the relative price of intertemporal consumption (the slopes of the indifference curves and budget constraint are equal). It is worth remembering that the indifference curve's slope measures the extra consumption that would be necessary in the following period to offset the loss of a unit of consumption in the current period. In contrast, the budget constraint's slope determines the premium, R, for saving more (Barro, 1997). It can be seen that high values for β (patient individuals) lead to indifference curves having low slopes.

A rise in R reduces the next period's cost of consumption, relative to current consumption, because of to households having the possibility of obtaining more future units of consumption for each previous unit of current consumption. It is this change in relative price that motivates households to increase future consumption in relation to present consumption. Economists call this mechanism the intertemporal substitution effect (Barro, 1997).

Figure 2.10 shows a graphical example, in which an individual's optimal choice is c_1^* in period 1 and c_2^* in period 2 (point E). It also shows an individual's income in both periods. What is actually shown is both periods' real income ($\frac{Y_1}{P_1}$ and $\frac{Y_2}{P_2}$) in which consumption levels are equal to income levels for the two periods (point D). Analyzing figure 2.10, optimal consumption in period (c_1^*) is greater than real income in the same period ($\frac{Y_1}{P_1}$), indicating that the individual is less patient in relation to current consumption. This individual is spending more than he/she earns, meaning that part of his/her wealth must be used to cover the period's excess consumption. Mathematically, rearranging period 1 budget constraint,

$$c_1 - \frac{Y_1}{P_1} = -\frac{A_1}{P_1}$$

it can be seen that if $c_1 > \frac{Y_1}{P_1}$ real wealth in period 1 will be negative, $-\frac{A_1}{P_1}$, indicating that this individual is a borrower. To see the consequences of this, period 2 budget constraint will be altered in the following way:

$$c_2 - \frac{Y_2}{P_2} = \frac{RA_1}{P_2}$$

as A_1 is negative (the individual is a borrower), the left-hand side of the previous expression should also be negative, $c_2 < \frac{Y_2}{P_2}$, indicating that consumption in period 2 should be lower than real income in this period. The reason for this occurrence lies in the fact that the individual has to pay off the loan arranged in period 1. Thus, consumption higher than real income in period 1 must be balanced with real income higher than consumption in period 2.

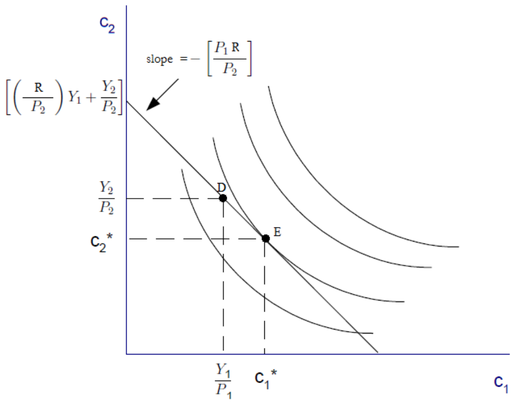

Figure 2.10: Interaction between the intertemporal budget constraint and an individual's preference to determine optimal intertemporal consumption.

Input markets

Firms represent the agents that acquire inputs, while households are those that supply them. Adjustments in input markets determine the amount of inputs and the aggregate product of an economy. In this section, we will explore how each input's price level is determined.

Definition of input markets

Generally speaking, it is assumed that inputs are physically equal. Households sell labor on the labor market and rent capital on the capital market. These markets establish unique price and wage (W) levels, and a unique return on capital (R) level, for the labor and capital markets, respectively. Thus, firms hire labor and capital paying W and R monetary units per hour, respectively. At the same time, the suppliers of inputs receive W and R monetary units for each hour of service supplied. It is assumed that firms and households take input price levels as a given.

Thus, L^s and L^d are the number of working hours households supply and the number of working hours firms demand in the labor market in each period. K^s and K^d are, respectively, the number of hours of rent of capital that households supply and that firms demand in the capital market in each period. All firms use inputs to produce goods using a production function:

$$Y = f(K^d, L^d)$$

This production function can be represented on a graph by an isoquant curve, a contour line showing the combinations of capital and labor that generate the same level of production. Plotting isoquant curves on a graph results in an isoquant map (figure 2.11).

Provided that the goods produced are being sold at price P, a firm's profit can be defined by:

$$\text{Profit} = PY - WL^d - RK^d$$

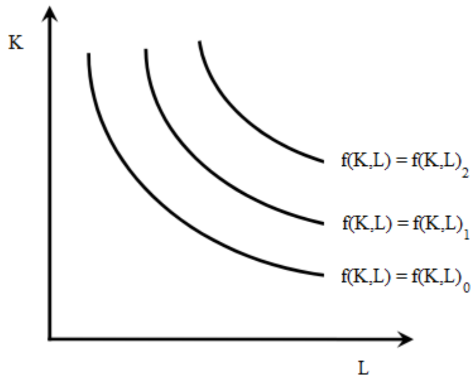

Figure 2.11: Isoquant map.

Demand for inputs

Firms define their demand for inputs aiming to maximize profits in each period.

$$\max_{K^d, L^d} PY - WL^d - RK^d$$

subject to the following technology:

$$Y = f(K^d, L^d)$$

The first-order conditions are:

$$P\frac{\partial Y}{\partial L^d} - W = 0$$

$$P\frac{\partial Y}{\partial K^d} - R = 0$$

Definition 2.1.5 (Problem of the firm)**.** *A profit-maximizing firm chooses input and production levels with the sole objective of maximizing economic profit. That is, firms wish to obtain the largest possible difference between total revenue and total costs.*

The first-order conditions of the problem of the firm may be written in the following manner:

Theoretical result 2.1.3 (Demand for Inputs)**.**

$$\underbrace{\frac{\partial Y}{\partial L^d}}_{\textit{Marginal Productivity of Labor (MPL)}} = \underbrace{\frac{W}{P}}_{\textit{Real Marginal Cost of Labor (real MCL)}}$$

$$\underbrace{\frac{\partial Y}{\partial K^d}}_{\textit{Marginal Productivity of Capital (MPK)}} = \underbrace{\frac{R}{P}}_{\textit{Real Marginal Cost of Capital (real MCK)}}$$

The theoretical result 2.1.3 states that firms choose input levels so that the marginal product of these inputs equals their real marginal costs. At this point, the last unit of an input contributes to the product enough to cover the extra cost of this unit of input in units of goods.

Combining the last two expressions:

Theoretical result 2.1.4 (Relative Demand for Inputs).

$$\underbrace{-\frac{\frac{\partial Y}{\partial L^d}}{\frac{\partial Y}{\partial K^d}}}_{\textit{Marginal Rate of Technical Substitution (MRTS)}} = \underbrace{-\frac{W}{R}}_{\textit{Economic Rate of Substitution (ERS)}}$$

Definition 2.1.6 (Marginal Rate of Technical Substitution). *The negative slope of an isoquant curve consisting of two inputs, capital (K) and labor (L) is called the Marginal Rate of Technical Substitution (MRTS) at that point. That is,*

$$MRTS = -\left.\frac{\partial K}{\partial L}\right|_{f(K,L)=f(K,L)_1} = -\left.\frac{PMgL}{PMgK}\right|_{f(K,L)=f(K,L)_i}$$

where $|_{f(K,L)=f(K,L)_i}$ *indicates the slope is calculated along the isoquant* $f(K,L)_i$.

Intuitively, the marginal rate of technical substitution indicates how many additional units of capital should be employed to offset one less unit of labor.

Summarizing the results obtained in this section, theoretical results 2.1.1, 2.1.2, and 2.1.4 show the same features. Agents, when deciding upon their choices, use marginal rates of substitution between goods and their relative prices. First, households must face the consumption-leisure tradeoff analyzing the relative price between these goods (real wages). When the choice is intertemporal, the tradeoff is between consumption today and future consumption, and the relative price is the nominal interest rate. Firms must make the same type of decision when deciding the combination of units of labor and capital to be used, analyzing the relative prices of these inputs (W/R).

The model

In this section, the structural model of the economy proposed in this chapter is presented and solved, step by step. This begins with the presentation of the agents (households and firms), following which the equilibrium conditions are shown. Then, the steady state is found and the equations that make up the model's equilibrium are log-linearized.

Assumption 2.2.1. *The economy is closed, with no government or financial sector.*

Assumption 2.2.2. *This economy does not have a currency. That is, it is a barter economy.*

Assumption 2.2.3. *Adjustment costs do not exist.*

Households

Assumption 2.2.4. *The economy in this model is formed by a unitary set of households indexed by* $j \in [0,1]$ *whose problem is to maximize a particular intertemporal welfare function. To this end, a utility function is used, additively separable into consumption (C) and labor (L).*

It is to be expected that a rise in consumption brings utility (happiness) to households, while a rise in labor hours brings disutility. At this point in the book, this is not surprising, seeing as in the theoretical section, it was mentioned that leisure provides individuals with happiness and that the more time they spend working, the less time they will have for leisure.

Assumption 2.2.5. *Consumption is intertemporally additively separable (no habit formation).*

Assumption 2.2.6. *Population growth is ignored.*

Assumption 2.2.7. *The labor market structure is one of perfect competition (no wage rigidity).*

The representative household optimizes the following welfare function:

$$\max_{C_{j,t},L_{j,t},K_{j,t+1}} E_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{j,t}^{1-\sigma}}{1-\sigma} - \frac{L_{j,t}^{1+\varphi}}{1+\varphi} \right) \quad (2.1)$$

where E_t is the expectations operator, β is the intertemporal discount factor, C is the consumption of goods, L is the number of hours worked, σ is the relative risk aversion coefficient, and φ is the marginal disutility in respect of labor supply.

As mentioned in the theoretical section, the utility function[8] must have certain characteristics: $U_C > 0$ and $U_L < 0$, this means that consumption and labor have positive and negative effects, respectively, on the utility of households. On the other hand, $U_{CC} < 0$ and $U_{LL} < 0$ indicate that the utility function is concave[9]. This represents the fact that, as consumption increases, so does utility, albeit at increasingly lower rates.

Households maximize their welfare function, which is subject to their intertemporal budget constraints, which indicates which resources are available and how they are allocated. Thus, it is assumed that households are the owners of the economy's factors of production (capital and labor). Households, providing labor and capital to firms, receive wages and returns on capital, respectively. They also own the firms, and therefore receive dividends. Thus, a household's intertemporal budget constraint can be written in the following way:

$$P_t(C_{j,t} + I_{j,t}) = W_t L_{j,t} + R_t K_{j,t} + \Pi_t \quad (2.2)$$

where P is the general price level, I is level of investment, W is the level of wages, K is the capital stock, R is the return on capital, and Π is the firms' profit (dividends).

---

[8] The most common utility function used to represent Household choices is the utility function with a constant relative risk aversion (CRRA) (Galí, 2008; Lim and McNelis, 2008; Clarida *et al.*, 2002; Galí and Monacelli, 2005; Christoffel and Kuester, 2008; Christoffel *et al.*, 2009; Ravenna and Walsh, 2006, among others). In the literature, other functions that represent utility do exist, for example: a logarithmic utility function, $U(C_t, L_t) = \ln C_t + \frac{L_t}{L_0} A \ln(1 - L_0)$ (Hansen, 1985); a utility function that is a combination of a logarithmic function and CRRA, $U(C_t, L_t) = \ln(C_t) - \frac{\upsilon}{1+\chi} L_t^{1+\chi}$ (Gertler and Karadi, 2011, among others).

[9] U_C and U_L are the first-order derivatives of the utility function in relation to consumption and labor, respectively, while, U_{CC} and U_{LL} are the second-order derivatives.

An additional equation that shows capital accumulation over time is required.

$$K_{j,t+1} = (1-\delta)K_{j,t} + I_{j,t} \tag{2.3}$$

where δ is the depreciation rate of physical capital.

The problem of the household is solved using the following Lagrangian formed by equations (2.1), (2.2) and (2.3):

$$\mathscr{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \left[\frac{C_{j,t}^{1-\sigma}}{1-\sigma} - \frac{L_{j,t}^{1+\varphi}}{1+\varphi} \right] \right.$$

$$\left. -\lambda_{j,t} \left[P_t C_{j,t} + P_t K_{j,t+1} - P_t(1-\delta)K_{j,t} - W_t L_{j,t} - R_t K_{j,t} - \Pi_t \right] \right\} \tag{2.4}$$

where λ is the Lagrange multiplier.

Solving the previous problem, we arrive at the following first-order conditions:

$$\frac{\partial \mathscr{L}}{\partial C_{j,t}} = C_{j,t}^{-\sigma} - \lambda_{j,t} P_t = 0 \tag{2.5}$$

$$\frac{\partial \mathscr{L}}{\partial L_{j,t}} = -L_{j,t}^{\varphi} + \lambda_{j,t} W_t = 0 \tag{2.6}$$

$$\frac{\partial \mathscr{L}}{\partial K_{j,t+1}} = -\lambda_{j,t} P_t + \beta E_t \lambda_{j,t+1} \left[(1-\delta) E_t P_{t+1} + E_t R_{t+1} \right] = 0 \tag{2.7}$$

Solving for λ_t equations (2.5) and (2.6), we arrive at the household's labor supply equation.

$$C_{j,t}^{\sigma} L_{j,t}^{\varphi} = \frac{W_t}{P_t} \tag{2.8}$$

or,

$$\underbrace{-C_{j,t}^{\sigma} L_{j,t}^{\varphi}}_{\text{Consumption-leisure MRS}} = \underbrace{-\frac{W_t}{P_t}}_{\text{Consumption-leisure relative price}}$$

The labor supply equation states that the consumption-leisure relative price (real wage) must be equal to the leisure-consumption marginal rate of substitution (Theoretical Result 2.1.1). Thus, a rise in consumption, ceteris paribus, is only possible with a rise in the amount of labor hours (less leisure). In other words, there is a trade-off between working less (enjoying less leisure) and consuming more. On the other hand, higher real wages allow consumption to increase without there being a need to give up leisure[10].

Knowing that from equation (2.5) $\lambda_{j,t} = \frac{C_{j,t}^{-\sigma}}{P_t}$ e $\lambda_{j,t+1} = \frac{C_{j,t+1}^{-\sigma}}{P_{t+1}}$, and using these results in equation (2.7), the Euler equation is found:

$$-C_{j,t}^{-\sigma} + \beta E_t \left\{ \left(\frac{C_{j,t+1}^{-\sigma}}{P_{t+1}} \right) [(1-\delta)P_{t+1} + R_{t+1}] \right\} = 0$$

$$\left(\frac{E_t C_{j,t+1}}{C_{j,t}} \right)^{\sigma} = \beta \left[(1-\delta) + E_t \left(\frac{R_{t+1}}{P_{t+1}} \right) \right] \tag{2.9}$$

The previous equation determines the household's savings decision (in this model, savings is the acquisition of investment goods). Thus, when households decide their level of savings, they compare the utility rendered by consuming an additional amount today with the utility that would be rendered by consuming more in the future. Thus, if interest rate expectations rise, consuming "today" (at t) is more expensive and, ceteris paribus, future consumption (t+1) will rise.

One final remark concerning the Euler equation is worth being made. To simplify it, assume that $\beta = 1$ and $\delta = 1$,

$$\underbrace{-E_t \left[\frac{1}{\pi_{t+1}} \left(\frac{C_{j,t+1}}{C_{j,t}} \right)^{\sigma} \right]}_{\text{TMS } C_t\text{-}C_{t+1}} = \underbrace{-E_t \left(\frac{r_{t+1}}{\pi_{t+1}} \right)}_{\text{relative price } C_t\text{-}C_{t+1}}$$

where $E_t r_{t+1} = E_t \left(\frac{R_{t+1}}{P_{t+1}} \right)$ is the real rate of return on capital.

Thus, this last expression (Theoretical Result 2.1.2) states that the marginal rate of substitution of current consumption for future consumption is equal to the relative price of current consumption in terms of future consumption.

[10] With higher real wages, the consumption of goods will certainly be higher. On the other hand, the same cannot be said for leisure. If the income effect exceeds the substitution effect, leisure will increase, however, in the opposite case, leisure will decrease.

In short, the problem of the household boils down to two choices. The first is an intratemporal choice between acquiring consumption and leisure goods. The other is an intertemporal choice, in which the household must choose between present and future consumption.

Firms

The representative firm is the agent that produces the goods and services that will be either consumed or saved (and then transformed into capital) by households.

Assumption 2.2.8. *There is a continuum of firms indexed by j that maximize profit observing a structure of perfect competition, this means that their profits will be zero ($\Pi_t = 0$, for every t).*

To this end a Cobb-Douglas[11] production function is used:

$$Y_{j,t} = A_t K_{j,t}^{\alpha} L_{j,t}^{1-\alpha} \tag{2.10}$$

where A_t represents productivity, a variable that can be interpreted as the level of general knowledge about the "arts" of production available in an economy, Y_t is the product, and α is the elasticity of the level of production with respect to capital; α can also be thought of as the level of participation of capital in the productive process, whereas $(1-\alpha)$ would be the level of participation of labor. Similarly to the household's utility function, the production function must have certain properties: it must be strictly increasing ($F_K > 0$ and $F_L > 0$), strictly concave ($F_{KK} < 0$ e $F_{LL} < 0$), and twice differentiable. It is also assumed that the production function has constant returns to scale, $F(zK_t, zL_t) = zY_t$. This function must also satisfy the Inada conditions: $\lim_{K\to 0} F_K = \infty$; $\lim_{K\to\infty} F_K = 0$; $\lim_{L\to 0} F_L = \infty$; and $\lim_{L\to\infty} F_L = 0$.

[11] Although many DSGE models use Cobb-Douglas technology, there are alternatives. Another very popular function in the literature is the CES (Constant Elasticity of Substitution) function,

$$F(K_t, L_t) = \left[\alpha K_t^{\rho} + (1-\alpha) L_t^{\rho}\right]^{\frac{1}{\rho}}$$

where $\rho \in (-\infty, 1)$ is a parameter that determines the elasticity of substitution between two inputs.

The problem of the firm is solved by maximizing the Profit function, choosing the amounts of each input (L_t, K_t):

$$\max_{L_{j,t},K_{j,t}} \Pi_{j,t} = A_t K_{j,t}^{\alpha} L_{j,t}^{1-\alpha} P_{j,t} - W_t L_{j,t} - R_t K_{j,t} \tag{2.11}$$

Solving the previous problem, we arrive at the following first-order conditions:

$$\frac{\partial \Pi_{j,t}}{\partial K_{j,t}} = \alpha A_t K_{j,t}^{\alpha-1} L_{j,t}^{1-\alpha} P_{j,t} - R_t = 0 \tag{2.12}$$

$$\frac{\partial \Pi_{j,t}}{\partial L_{j,t}} = (1-\alpha) A_t K_{j,t}^{\alpha} L_{j,t}^{-\alpha} P_{j,t} - W_t = 0 \tag{2.13}$$

From equations (2.12) and (2.13):

$$\underbrace{\frac{R_t}{P_{j,t}}}_{\text{Real MCK}} = \underbrace{\alpha \frac{Y_{j,t}}{K_{j,t}}}_{\text{MPK}} \tag{2.14}$$

$$\underbrace{\frac{W_t}{P_{j,t}}}_{\text{Real MCL}} = \underbrace{(1-\alpha) \frac{Y_{j,t}}{L_{j,t}}}_{\text{MPL}} \tag{2.15}$$

Equations (2.14) and (2.15) represent the demand for capital and labor, respectively (Theoretical Result 2.1.3), in which marginal costs are equal to the marginal products[12].

Note that in equation (2.15) a reduction in real wages means higher demand for labor as, when the real cost of hiring workers reduces, firms increase their demand for labor until the marginal product of labor reduces to the same level as the fall in real wages[13] (Barro, 1997).

It is assumed that productivity shocks follow a first-order autoregressive process, such that:

$$\log A_t = (1-\rho_A) \log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t \tag{2.16}$$

[12]Real MCK is the real marginal cost of capital;Real MCL is the real marginal cost of labor; MPK is the marginal product of capital; and MPL is the marginal product of labor.

[13]The same logic applies to capital (equation 2.14).

where A_{ss} is the value of productivity at the steady state, ρ_A is the autoregressive parameter of productivity, whose absolute value must be less than one ($|\rho_A| < 1$) to ensure the stationary nature of the process and $\epsilon_t \sim N(0, \sigma_A)$.

Assumption 2.2.9. *Productivity growth is ignored in this model.*

As the model follows the RBC approach, the price level must be equal to marginal cost. Thus, to obtain the marginal cost, the input demand equations must first be combined (equations (2.14) and (2.15)):

$$-\underbrace{\frac{W_t}{R_t}}_{\text{ERS}} = -\underbrace{\frac{(1-\alpha)K_{j,t}}{\alpha L_{j,t}}}_{\text{MRTS}}$$

Reminding the reader that this expression represents Theoretical Result 2.1.4. Its right-hand side is the marginal rate of technical substitution, which measures the rate at which labor can be replaced by capital while maintaining a constant level of production. The left-hand side is the economic rate of substitution, which measures the rate at which labor can be replaced by capital while maintaining the same cost.

Rearranging the previous expression,

$$L_{j,t} = \left(\frac{1-\alpha}{\alpha}\right)\frac{R_t}{W_t}K_{j,t} \tag{2.17}$$

and substituting equation (2.17) in the production function (equation (2.10)),

$$Y_{j,t} = A_t K_{j,t}^{\alpha}\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{R_t}{W_t}K_{j,t}\right]^{1-\alpha}$$

$$K_{j,t} = \frac{Y_{j,t}}{A_t}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{1-\alpha} \tag{2.18}$$

Substituting equation (2.18) in (2.17),

$$L_{j,t} = \frac{Y_{j,t}}{A_t}\left(\frac{1-\alpha}{\alpha}\right)\frac{R_t}{W_t}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{1-\alpha}$$

$$\left(\frac{1-\alpha}{\alpha}\right)\frac{R_t}{W_t} = \left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{-1}$$

$$L_{j,t} = \frac{A_t}{Y_{j,t}}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{-\alpha} \tag{2.19}$$

total cost (TC) is represented by:

$$TC_{j,t} = W_t L_{j,t} + R_t K_{j,t}$$

substituting equations (2.18) and (2.19) in the total cost function:

$$TC_t = W_t\frac{Y_{j,t}}{A_t}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{-\alpha} + R_t\frac{Y_{j,t}}{A_t}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{1-\alpha}$$

with a little algebraic massaging, we arrive at:

$$TC_{j,t} = \frac{Y_{j,t}}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha}$$

and the marginal cost is derived from the total cost[14]:

$$MC_{j,t} = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha} \tag{2.20}$$

As the marginal cost depends solely on productivity and the prices of the factors of production, it will be the same for all firms ($MC_{j,t} = MC_t$). Knowing that $P_t = MC_t$, we arrive at the general price level,

$$P_t = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha} \tag{2.21}$$

The model's equilibrium conditions

Now that each agent's behavior has been described, the interaction between them must be studied in order to determine macroeconomic equilibrium. Households decide how much to consume (C), how much to invest (I) and how much labor to supply (L), with the aim of maximizing utility, taking prices as given. On the other

[14] $MC_{j,t} = \frac{\partial TC_{j,t}}{\partial Y_{j,t}}$.

hand, firms decide how much to produce (Y) using available technology and choosing the factors of production (capital and labor), taking these prices as given.

Therefore, the model's equilibrium consists of the following blocks:

1. a price system, W_t, R_t and P_t;
2. an endowment of values for goods and inputs Y_t, C_t, I_t, L_t and K_t; and
3. a production-possibility frontier described by the following equilibrium condition of the goods market (aggregate supply = aggregate demand).

$$Y_t = C_t + I_t \tag{2.22}$$

Competitive equilibrium consists in finding a sequence of endogenous variables in the model such that the conditions that define equilibrium are satisfied. In short, this economy's model consists of the following equations from Table 2.1[15].

Steady state

After defining the economy's equilibrium, the steady state values must be defined. Indeed, the model presented is steady in the sense that there exists a value for the variables that is maintained over time: an endogenous variable x_t is at the steady state in each t, if $E_t x_{t+1} = x_t = x_{t-1} = x_{ss}$.

Some endogenous variables have their steady state values previously determined (exogenously). This is the case of productivity, which is the source of standard RBC models' shocks - at the steady state $E(\epsilon_t) = 0$. Thus, with equation (2.16) it is not possible to know the value of productivity at the steady state, the literature generally assigning $A_{ss} = 1$. The next step is to remove the variables' time indicators. Therefore, the structural model is:

[15] Because of the symmetry in the preferences of households and in the technology of firms, these two kinds of agents will be represented by representative agents (this eliminates the j subscript).

Table 2.1: Structure of the model.

Equation	(Definition)
$C_t^\sigma L_t^\varphi = \frac{W_t}{P_t}$	(Labor supply)
$\left(\frac{E_t C_{j,t+1}}{C_{j,t}}\right)^\sigma = \beta\left[(1-\delta) + E_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right]$	(Euler equation)
$K_{t+1} = (1-\delta)K_t + I_t$	(Law of motion of capital)
$Y_t = A_t K_t^\alpha L_t^{1-\alpha}$	(Production function)
$K_t = \alpha \frac{Y_t}{\frac{R_t}{P_t}}$	(Demand for capital)
$L_t = (1-\alpha)\frac{Y_t}{\frac{W_t}{P_t}}$	(Demand for labor)
$P_t = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^\alpha$	(Price level)
$Y_t = C_t + I_t$	(Equilibrium condition)
$\log A_t = (1-\rho_A)\log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t$	(Productivity shock)

Households

$$C_{ss}^\sigma L_{ss}^\varphi = \frac{W_{ss}}{P_{ss}} \tag{2.23}$$

$$1 = \beta\left(1 - \delta + \frac{R_{ss}}{P_{ss}}\right) \tag{2.24}$$

$$I_{ss} = \delta K_{ss} \tag{2.25}$$

Firms

$$K_{ss} = \alpha \frac{Y_{ss}}{\frac{R_{ss}}{P_{ss}}} \tag{2.26}$$

$$L_{ss} = (1-\alpha)\frac{Y_{ss}}{\frac{W_{ss}}{P_{ss}}} \tag{2.27}$$

$$Y_{ss} = K_{ss}^\alpha L_{ss}^{1-\alpha} \tag{2.28}$$

$$P_{ss} = \left(\frac{W_{ss}}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_{ss}}{\alpha}\right)^\alpha \tag{2.29}$$

Equilibrium Condition

$$Y_{ss} = C_{ss} + I_{ss} \tag{2.30}$$

The system of equations formed by equations (2.23) to (2.30) will be used to determine the value of eight endogenous variables at the steady state (Y_{ss}, C_{ss}, I_{ss}, K_{ss}, L_{ss}, W_{ss}, R_{ss} and P_{ss}).

The first values that must be determined are the prices (W_{ss}, R_{ss} and P_{ss}). To this end, Walras' law must be taken into consideration.

Proposition 2.2.1 (Walras' Law)**.** *For any price vector **p**, has* $\mathbf{pz}(\mathbf{p}) \equiv 0$*; i.e., the demand excess value is identically zero.*

Proof. In simple terms, the definition of excess demand is written and multiplied by **p**:

$$\mathbf{pz}(\mathbf{p}) = \mathbf{p}\left[\sum_{i=1}^{n} \mathbf{x}_i(\mathbf{p}, \mathbf{p}\,\mathbf{w}_i) - \sum_{i=1}^{n} \mathbf{w}_i\right] = \sum_{i=1}^{n}\left[\mathbf{p}\,\mathbf{x}_i(\mathbf{p}, \mathbf{p}\,\mathbf{w}_i) - \mathbf{p}\,\mathbf{w}_i\right] = 0$$

since $\mathbf{x}_i(\mathbf{p}, \mathbf{p}\,\mathbf{w}_i)$ satisfies the budget constraint $\mathbf{p}\,\mathbf{x}_i = \mathbf{p}\,\mathbf{w}_i$ for each individual i=1,...,n. □

In other words, Walras' law states that if each individual satisfies his/her budget constraint, the value of his/her excess demand is zero, therefore the sum of excess demand must also be zero. It is important to note that Walras' law states that the value of excess demand is identical to zero - the value of excess demand is zero for all prices (Varian, 1992).

Walras' Law implies the existence of k-1 independent equations in equilibrium with k goods. Thus, if demand is equal to supply in k-1 markets, they will also be equal in the k^{th} market. Consequently, if there are k markets, only k-1 relative prices are required to determine equilibrium.

Provided that the excess aggregate demand function is homogeneous of degree zero, prices can be normalized and demands expressed in terms of relative price: $p_i = \frac{\hat{p}_i}{\sum_{j=1}^{k} \hat{p}_j}$. As a consequence, the sum of the normalized prices p_i must always be 1. Thus, attention can be directed to the price vector belonging to the unit simplex of dimension k-1: $S^{k-1} = \left\{\mathbf{p} \in R_+^k : \sum_{i=1}^{k} p_i = 1\right\}$. In short, taking Walras' Law into account, the economy's general price level can be normalized, $P_{ss} = 1$.

To find R_{ss}, equation (2.24) is used,

$$R_{ss} = P_{ss}\left[\left(\frac{1}{\beta}\right) - (1-\delta)\right] \tag{2.31}$$

Note that equation (2.31) shows R_{ss} as a function of only the normalized general price level parameters[16], therefore its value is determined. It simply remains to find the steady state of the wage level (W_{ss}). Thus, from equation (2.29),

$$W_{ss}^{1-\alpha} = P_{ss}(1-\alpha)^{1-\alpha}\left(\frac{\alpha}{R_{ss}}\right)^{\alpha}$$

$$W_{ss} = (1-\alpha)P_{ss}^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{R_{ss}}\right)^{\frac{\alpha}{1-\alpha}} \tag{2.32}$$

The next step is to satisfy the equilibrium condition. To this end, the variables that make up aggregate demand (C_{ss} and I_{ss}) must be determined. The idea underlying the equilibrium condition is formed by the following proposition.

Proposition 2.2.2 (Market adjustment)**.** *Given k markets, if demand is equal to supply in k-1 markets and* $p_k > 0$*, then demand must equal supply in the* k^{th} *market.*

Proof. If not, Walras' Law is violated. □

Therefore, to meet the equilibrium condition, the input market conditions must be met. To this end, it is necessary to find the meeting point between the supplies (provided by households) and the demands (provided by firms) of the production inputs (labor and capital) (Figure 2.12).

First, equation (2.27) must be replaced in equation (2.23), solving for C_{ss},

$$C_{ss}^{\sigma}\left[(1-\alpha)\frac{Y_{ss}}{\frac{W_{ss}}{P_{ss}}}\right]^{\varphi} = \frac{W_{ss}}{P_{ss}}$$

[16] In the Dynare simulation, there is no need to substitute R_{ss} in the other equations. It should just be shown before the other steady states.

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